6D Vlasov-Poisson Simulation of Self-Gravitating Systems and Its Application to Dynamics of Cosmic Neutrinos

Vlasov-Poisson : towards numerical methods without particles

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Numerical Simulation of Collisionless Self-Gravitating Systems

- N-body simulations

  - a “de facto standard” method to simulate the nonlinear evolution of self-gravitating systems for more than 30 years.
  
  - the mass distribution is sampled by particles in the 6D phase-space volume in a Monte-Carlo manner.
  
  - very large number of particles can be treated with the aid of sophisticated Poisson solvers such as Tree and TreePM methods.

Trillion-body simulation by Ishiyama (2012)
Potential Drawbacks of N-body Simulations

- monte-carlo sampling of matter distribution by discrete super particles
  - intrinsic contamination of shot noise in physical quantities

- artificial two-body relaxation due to the super-particle approximation
  - introduces undesired collisional effect in a long-term evolution

- velocity space is rather sparsely sampled.
  - physical processes sensitive to the velocity structure such as the collisionless damping and the two-stream instability are not properly solved.
Past and Present of Simulations of Large-Scale Structure in the Universe

Miyoshi & Kihara (1975) PASJ, 27, 333

$N=400$ on HITAC 8500

Trillion-body simulation by Ishiyama (2012)

$N=10^{12}$ on K-computer

13.7 Gyr

$z=0$
Vlasov-Poisson Equations

\[
\begin{aligned}
\frac{\partial f}{\partial t} + \vec{v} \cdot \frac{\partial f}{\partial \vec{x}} - \nabla \phi \cdot \frac{\partial f}{\partial \vec{v}} &= 0 \\
\nabla^2 \phi &= 4\pi G \rho = 4\pi G \int f \, d^3 \vec{v}
\end{aligned}
\]

\( f(\vec{x}, \vec{v}) \) : matter density in 6D phase space (distribution function)

- combination of collisionless Boltzmann equation (aka Vlasov equation) and Poisson equation.

- treats the matter as continuum fluid in the phase space instead of statistically sampled particles

  - free from shot noise contamination seen in the N-body approach

- so far limited to 1D or 2D simulations due to the large amount of required memory space and huge computational costs.

We present the first 3D Vlasov-Poisson simulation in the 6D phase space volume.

KY, N. Yoshida, M. Umemura (2013)
Both of physical and velocity spaces are discretized with 3D regular mesh grids.

Vlasov equation is solved using directional splitting scheme, in which following 1D advection equations are sequentially integrated.

\[
\frac{\partial f}{\partial t} + v_i \frac{\partial f}{\partial x_i} = 0, \quad \frac{\partial f}{\partial t} - \frac{\partial \phi}{\partial x_i} \frac{\partial f}{\partial v_i} = 0 \quad (i = 1, 2, 3)
\]

Physical requirements for the scheme of 1D advection equations

- positivity
- mass conservation
- maximum principle

Positive Flux Conservation (PFC) scheme

Positive Flux Conservative Scheme

Positive Flux Conservative (PFC) method

\[ t = t^n + \Delta t \]

\[ t = t^n \]

\[ \int_{x_i-\Delta x/2}^{x_i+\Delta x/2} f(t^{n+1}, x) \, dx = \int_{x_{i-1}-\Delta x/2}^{x_{i-1}+\Delta x/2} f(t^n, x) \, dx + \Phi^+ - \Phi^- \]

\[ f_i^{n+1} = f_i^n + \frac{\Phi^+}{\Delta x} - \frac{\Phi^-}{\Delta x} \]

- PFC scheme satisfies all of the positivity, mass conservation and maximum principle
- effectively 3rd-order accuracy in space
Parallelization

- Only physical (spatial) grids are decomposed among computational nodes.
- Each spatial grids contains the entire velocity (momentum) grids.
- Data communication at the boundaries of each decomposed domain.
Time Integration

2\textsuperscript{nd} order leapfrog scheme with directional splitting

\begin{equation}
    f(\vec{x}, \vec{v}, t^{n+1}) = T_{v_x}(\Delta t/2)T_{v_y}(\Delta t/2)T_{v_z}(\Delta t/2)
    \begin{align*}
        T_x(\Delta t) & \cdot T_y(\Delta t) & \cdot T_z(\Delta t) \\
        T_{v_x}(\Delta t/2) & \cdot T_{v_y}(\Delta t/2) & \cdot T_{v_z}(\Delta t/2)
    \end{align*}
    f(\vec{x}, \vec{v}, t^n)
\end{equation}

After solving advection equations in the position space, we solve the Poisson equation to update the gravitational potential.

Timestep constraints

\begin{equation}
    \Delta t = C \min(\Delta t_v, \Delta t_x)
\end{equation}

\begin{align*}
    \Delta t_x &= \min \left( \frac{\Delta x}{V_x^{\max}}, \frac{\Delta y}{V_y^{\max}}, \frac{\Delta z}{V_z^{\max}} \right) \\
    \Delta t_v &= \min_i \left( \frac{\Delta v_x}{|a_{x,i}|}, \frac{\Delta v_y}{|a_{y,i}|}, \frac{\Delta v_z}{|a_{z,i}|} \right)
\end{align*}
Numerical Test Suite

- Stability test of a stable solution of Vlasov-Poisson equations
- Dynamical collision of two self-gravitating systems
- Gravitational instability and collisionless damping in homogeneous matter
Computational Resources

- Cray XC-30 system in NAOJ (National Astronomical Observatory Japan.)
  - Intel Xeon 2.6GHz x 20 cores + 128 GB RAM / node
  - 1050 nodes in total

Memory requirements for $N_x$ and $N_v$ mesh grids in position and velocity space

- $N_x=64^3$, $N_v=64^3$: 256 GB
- $N_x=128^3$, $N_v=64^3$: 2048 GB → 32 or 64 nodes
- $N_x=256^3$, $N_v=64^3$: 16 TB → 256 or 512 nodes

5 min / step with 64 nodes
King sphere

- a stable solution of Vlasov – Poisson equations

\[
f(E, t = 0) = \frac{\rho_0}{(2\pi\sigma^2)^{3/2}} \exp\left[(-E/\sigma^2) - 1\right] \quad E < 0
\]

\[
= 0 \quad E > 0
\]

\[E = \frac{1}{2} v^2 + \Phi\]

- number of mesh points
  - \(64^3\) for the physical space
  - \(64^3\) or \(32^3\) for the velocity space

- isolated boundary condition
King Sphere

- kinetic and grav. potential energies are almost constant over the dynamical timescale.
- time variation of total, kinetic, and grav. potential energies is sufficiently small (not larger than 1%).
- the total mass is also conserved with sufficiently good accuracy.
King Sphere

time evolution of the King sphere

the profiles almost keep still.

slight mass transfer from center to outskirts, probably due to poor spatial resolution in the central region.
Merging of Two King Spheres

- initial condition
- offset merging of two King spheres
- $64^3$ mesh points for both the physical and velocity spaces.

N-body simulation for the comparison

- each King sphere is represented with $10^6$ particles
- solved using the Particle–Mesh method with the same spatial resolution as the Vlasov – Poisson simulation.
Merging of Two King Spheres

Vlasov simulation

N-body simulation
Merging of Two King Spheres

time evolution of the kinetic, grav. potential and total energy in Vlasov – Poisson and N-body simulations

good agreement between Vlasov – Poisson simulations and N-body simulations.

energy conservation is kept within 1% error at t<4.5T.
Velocity Distribution

phase space density in the central region at a time of the closest approach.

Vlasov – Poisson simulation

N-body simulation
Velocity Distribution

phase space density in the outskirts at a time of the closest approach.

Vlasov – Poisson simulation

N-body simulation
3D Gravitational Instability and Collisionless Damping

**Initial condition**

\[
\begin{align*}
    f(\vec{x}, \vec{v}, t = 0) &= \bar{\rho}(1 + \delta(x)) \frac{(1 + \delta(x))}{(2\pi\sigma^2)^{3/2}} \exp \left( -\frac{\mid\vec{v}\mid}{2\sigma^2} \right) \\
    \rho(x, t = 0) &= \bar{\rho}(1 + \delta(x))
\end{align*}
\]

- The density fluctuation $\delta(x)$ is given so that it has a white noise power spectrum.
- periodic boundary condition
- Jeans wave number

\[
k_J = \sqrt{\frac{4\pi G \bar{\rho}}{\sigma}}
\]

- number of mesh grids
  - $64^3$ for the physical space
  - $64^3$ or $32^3$ for the velocity space

- $k > k_J$ $\rightarrow$ collisionless damping
- $k < k_J$ $\rightarrow$ gravitational instability
3-D Self-Gravitating System

t=0.2T

t=0.4T

t=0.6T

- time evolution of the density map

- small scale fluctuation at the initial condition damps through the collisionless damping

- density fluctuations with a scale larger than Jeans length grow through the gravitational instability
Growth and damping of the density fluctuations switch each other clearly at the Jeans wave number.
Astrophysical Application

Prof. Tremaine gave me a list of problems which should be addressed with Vlasov-Poisson simulations:

- Adiabatic growth of a point mass in the center of a spherical cluster
- Onset of radial-orbit instability
- The normal modes of the Kalnajs disks
- The normal modes of some differentially rotating disks
- Adiabatic growth of a disk in a dark matter halo and $m=1$ instabilities in cusps around massive black holes

Dynamics of cosmic neutrinos in the large-scale structure formation
Neutrinos in Large-Scale Structure

Discovery of neutrino oscillation turns out that neutrinos are massive.

- dynamical effect on the large-scale structure formation

Ground-based experiments can only probe the mass difference between different flavors but not the absolute mass of neutrinos.

\[ 0.05 \text{ eV} \leq \sum_{i=1}^{3} m_i \leq 1.4 \text{ eV} \]

- The absolute mass and the mass hierarchy are important for theories beyond the standard model of elementary particles.

- The imprints of massive neutrinos on LSS in the universe is very important.
Collisionless Damping of Neutrinos

- very large velocity dispersion of neutrinos
  \[ \sigma_v = 150(1 + z) \left( \frac{1\,\text{eV}}{m_{\nu}} \right) \, \text{km/s} \]

- collisionless damping (free streaming)
  suppress the growth of density perturbation
  \[ \delta \propto D_+(t)^{1-\frac{3}{5}} f_{\nu} \]

  \( f_{\nu} \) : neutrino mass fraction

- scale of collisionless damping
  \[ k_{nr} = 0.018 \Omega_m^{1/2} \left( \frac{m_{\nu}}{1\,\text{eV}} \right) h \, \text{Mpc}^{-1} \]
  \[ \lambda_{nr} = 640 \left( \frac{\Omega_m}{0.3} \right)^{-1/2} \left( \frac{m_{\nu}}{1\,\text{eV}} \right)^{-1/2} h^{-1} \text{Mpc} \]

LSS observations can probe the absolute mass of neutrinos.
Simulation with CDM and neutrino

Previous simulations based-on N-body methods

- Dynamics of neutrinos are solved with the linear perturbation theory instead of being solved in a consistent manner
  
  Viel et al. (2010), Brandbyge & Hannestad (2009, 2010)

- Thermal velocities are assigned to particles for neutrinos statistically.
  
  e.g. Wagner, Verde, Jimenez (2012)

Dynamics of cosmic neutrinos as a good target of Vlasov-Poisson simulations.

- Collisionless damping of neutrinos are expected to be better simulated with V-P simulations.
Formulation in Comoving Frame

- Vlasov equation in comoving coordinate
\[
\frac{\partial f}{\partial t} + \frac{\vec{p}}{a^2} \cdot \frac{\partial f}{\partial \vec{x}} - \nabla \phi \cdot \frac{\partial f}{\partial \vec{p}} = 0 \quad \vec{p} = a^2 \vec{x}
\]

- Normalization of DF
\[
\int f d\nu^3 = 1 + \delta
\]

- Poisson equation
\[
\nabla^2 \phi = 4\pi G \bar{\rho} a^2 \left( \int f d\nu^3 - 1 \right)
\]
Formulation in Comoving Frame

Vlasov-Poisson equation with canonical variables in the comoving frame

\[
\frac{\partial f}{\partial t} + \frac{\vec{p}}{a^2} \cdot \frac{\partial f}{\partial \vec{x}} - \nabla \phi \cdot \frac{\partial f}{\partial \vec{p}} = 0 \quad \vec{p} = a^2 \dot{\vec{x}}
\]

\[
\nabla^2 \phi = 4\pi G a^2 \bar{\rho}\delta = 4\pi G a^2 \bar{\rho} \left[ \int f d^3 \vec{p} - 1 \right]
\]

- Extent of DF in the momentum space rapidly increases according to the expansion of the universe

modified Vlasov-Poisson equation in terms of peculiar velocity

\[
\frac{\partial f}{\partial t} + \frac{\vec{v}}{a} \cdot \frac{\partial f}{\partial \vec{x}} - \left[ H \vec{v} + \frac{\nabla \phi}{a} \right] \cdot \frac{\partial f}{\partial \vec{v}} = 0 \quad \vec{v} = a \dot{\vec{x}}
\]

- Extent of DF in this formulation does not change very much.

- We need a tricky scheme for the advection in the velocity space.
Cosmological Vlasov Simulation

ΛCDM universe with WMAP-9 yr cosmological parameters \( L = 120 \text{ Mpc}/h \)

Vlasov-Poisson simulation

\[ N_x = 128^3 \quad N_y = 64^3 \]

N-body simulation

\[ N_p = 128^3 \]
Power Spectra

consistent $P(k)$ between N-body and Vlasov simulations

Difference in $P(k)$ on smaller scales due to numerical diffusion
Simulations with CDM and Neutrinos

- hybrid of N-body and Vlasov-Poisson simulations

- cold dark matter: N-body simulation
  \[
  \frac{d^2 \vec{x}_i}{dt^2} + 2H \frac{d\vec{x}_i}{dt} = -\frac{1}{a^2} \nabla \phi
  \]

- neutrino (hot dark matter): Vlasov-Poisson simulation
  \[
  \frac{\partial f}{\partial t} + \frac{\vec{v}}{a} \cdot \frac{\partial f}{\partial \vec{x}} - \left[ H \vec{v} + \frac{\nabla \phi}{a} \right] \cdot \frac{\partial f}{\partial \vec{v}} = 0
  \]

- Poisson equation
  \[
  \nabla^2 \phi = 4\pi G \bar{\rho} a^2 (f_{\text{cdm}} \delta_{\text{cdm}} + f_\nu \delta_\nu)
  \]
Simulations with CDM and Neutrinos

\[ L = 2000h^{-1} \text{Mpc} \quad N_{\text{CDM}} = 128^3 \]
\[ N_x = 128^3 \quad N_\nu = 64^3 \text{ for Vlasov-Poisson simulation} \]
\[ \sum_i m^i_\nu = 3 \text{eV} \]

- density map of CDM and neutrino
Dumping of Power Spectrum

\[ L = 2000 h^{-1} \text{Mpc} \quad N_{\text{CDM}} = 128^3 \]

\[ N_\chi = 128^3 \quad N_\nu = 64^3 \text{ for Vlasov-Poisson simulation} \]

\[ \sum_i m^i_\nu = 3 \text{eV} \]
Vlasov-Poisson simulation in 6-dimensional phase space

Scientifically meaningful resolution (128³ x 64³ mesh) and accuracy can be achieved even by currently available computational resources

Independent and complementary alternative to the N-body approach

Application to the dynamics of cosmic neutrinos in cosmological context

- formulation and implementation in the cosmological comoving coordinate.
- hybrid of N-body and Vlasov-Poisson simulations
- comparison with N-body approaches required
Summary & Future Prospects

6D Vlasov simulations are always memory limited.

- Higher order schemes
- Schemes with adaptive or moving mesh techniques